

Chapter 7

Delamination/Interlaminar Crack Analysis

7.1 INTRODUCTION

7.2 FRACTURE MECHANICS FOR BIMATERIAL INTERFACE CRACKS

A traction free interface crack between two dissimilar orthotropic materials is considered, as depicted in Fig. 7.1. The crack is located in the (x,y) plane and (r,θ) are the local polar coordinates defined on the crack-tip. The elastic principal directions of each material are assumed to be normal to or parallel with the crack direction.

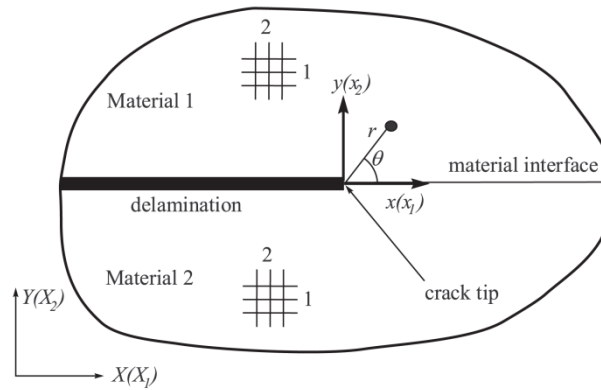


Figure 7.1 Interface crack between two orthotropic materials.

The general form of the asymptotic solution for a bimaterial interface crack can be written as,

$$u_i = \sum_j \{K_I A_j F_j(r, \theta) + K_{II} B_j F_j(r, \theta)\}, \quad i = x, y \quad (7.1)$$

where K_I and K_{II} are the mode 1 and 2 stress intensity factors, respectively, A_j and B_j are spatially-independent functions of the elasticity coefficients of the two materials, and F_j are functions of local crack tip coordinates (r, θ) , mainly representing the radial/angular nature of the displacement field.

In the following, two cases of isotropic and orthotropic bimetals are comprehensively examined.

7.2.1 Isotropic bimaterial interfaces

Deng (1995) developed the basic asymptotic solution for stationary and moving cracks with frictional contact along bimaterial interfaces in homogeneous solids. Neglecting the effects of crack propagation and contact, Sukumar *et al.* (2004) adopted the methods by Rice (1988), Suo (1990), Rice *et al.* (1990) and Shih (1991) to present the asymptotic solution for isotropic bimaterial cracks. For the material above the interface ($k = 1$),

$$\begin{aligned} u_x = & \frac{e^{-\varepsilon(\pi-\theta)} \operatorname{Re}(\bar{K}r)^{i\varepsilon}}{2\mu(1+4\varepsilon^2) \cosh(\pi\varepsilon)} \sqrt{\frac{r}{2\pi}} \left[-e^{2\varepsilon(\pi-\theta)} \left(\cos \frac{\theta}{2} + 2\varepsilon \sin \frac{\theta}{2} \right) \right. \\ & \left. + \kappa \left(\cos \frac{\theta}{2} - 2\varepsilon \sin \frac{\theta}{2} \right) + (1+4\varepsilon^2) \sin \frac{\theta}{2} \cos \theta \right] \\ & + \frac{e^{-\varepsilon(\pi-\theta)} \operatorname{Im}(\bar{K}r)^{i\varepsilon}}{2\mu(1+4\varepsilon^2) \cosh(\pi\varepsilon)} \sqrt{\frac{r}{2\pi}} \left[e^{2\varepsilon(\pi-\theta)} \left(\sin \frac{\theta}{2} - 2\varepsilon \cos \frac{\theta}{2} \right) \right. \\ & \left. + \kappa \left(\sin \frac{\theta}{2} + 2\varepsilon \cos \frac{\theta}{2} \right) + (1+4\varepsilon^2) \cos \frac{\theta}{2} \sin \theta \right] \end{aligned} \quad (7.2)$$

$$\begin{aligned} u_y = & \frac{e^{-\varepsilon(\pi-\theta)} \operatorname{Re}(\bar{K}r)^{i\varepsilon}}{2\mu(1+4\varepsilon^2) \cosh(\pi\varepsilon)} \sqrt{\frac{r}{2\pi}} \left[e^{2\varepsilon(\pi-\theta)} \left(\sin \frac{\theta}{2} - 2\varepsilon \cos \frac{\theta}{2} \right) \right. \\ & \left. + \kappa \left(\sin \frac{\theta}{2} + 2\varepsilon \cos \frac{\theta}{2} \right) - (1+4\varepsilon^2) \cos \frac{\theta}{2} \sin \theta \right] \\ & + \frac{e^{-\varepsilon(\pi-\theta)} \operatorname{Im}(\bar{K}r)^{i\varepsilon}}{2\mu(1+4\varepsilon^2) \cosh(\pi\varepsilon)} \sqrt{\frac{r}{2\pi}} \left[e^{2\varepsilon(\pi-\theta)} \left(\cos \frac{\theta}{2} + 2\varepsilon \sin \frac{\theta}{2} \right) \right. \\ & \left. - \kappa \left(\cos \frac{\theta}{2} - 2\varepsilon \sin \frac{\theta}{2} \right) + (1+4\varepsilon^2) \sin \frac{\theta}{2} \sin \theta \right] \end{aligned} \quad (7.3)$$

where κ is defined in (2.68) and $\bar{K}$ is the complex stress intensity factor,

$$\bar{K} = K_I + iK_{II} \quad (7.4)$$

and can be defined in a similar way to classical stress intensity factors,

$$\left[\sigma_{yy} + i\sigma_{xy} \right]_{\substack{r \rightarrow 0 \\ \theta = 0}} = \frac{\bar{K} r^{i\varepsilon}}{\sqrt{2\pi r}} \quad (7.5)$$

$\bar{K}$ can be associated with the energy release rate by,

$$G = \frac{1}{E^{12}} \frac{|\bar{K}|^2}{\cosh(\pi\varepsilon)} = \frac{1}{E^{12}} \frac{K_I^2 + K_{II}^2}{\cosh(\pi\varepsilon)} \quad (7.6)$$

where E^{12} is the equivalent bimaterial elastic modulus defined in terms of the effective Young's modulus E'_k ($k=1,2$) of each material (see (2.113))

$$\frac{2}{E^{12}} = \frac{1}{E'_1} + \frac{1}{E'_2} \quad (7.7)$$

ε is the index of oscillation defined as

$$\varepsilon = \frac{1}{2\pi} \ln \left(\frac{1-\beta}{1+\beta} \right) \quad (7.8)$$

where β is the second Dundurs parameter for isotropic materials (Dundurs, 1969)

$$\beta = \frac{\mu_1(\kappa_2 - 1) - \mu_2(\kappa_1 - 1)}{\mu_1(\kappa_2 + 1) + \mu_2(\kappa_1 + 1)} \quad (7.9)$$

The asymptotic solutions (7.2) and (7.3) can be used for points on the lower half plane by replacing $\varepsilon\pi$ with $-\varepsilon\pi$.

Another important parameter in characterization of interfacial fracture is the phase angle ψ , defined at a distance r from the crack tip,

$$\psi = \tan^{-1} \left[\frac{\text{Im}(\bar{K} r^{i\varepsilon})}{\text{Re}(\bar{K} r^{i\varepsilon})} \right] \quad (7.10)$$

The phase angle ψ is a measure of the relative proportion of shear to normal tractions at a distance r from the crack tip (Sukumar *et al.*, 2004).

7.2.2 Orthotropic bimaterial interface cracks

Lee *et al.* (1996) derived the governing equilibrium, elastic constitutive law and compatibility equations for orthotropic bimaterial interface cracks and combined them into a partial differential equation with the following characteristic equation in terms of a complex variable $z = x + my$,

$$m^4 + 2B_{12}m^2 + B_{66} = 0 \quad (7.11)$$

where

$$B_{12} = \frac{1}{2} \frac{(2c_{12} + c_{66})}{c_{11}} \quad (7.12)$$

$$B_{66} = \frac{c_{22}}{c_{11}} \quad (7.13)$$

The roots m_l and m_s of the characteristic equation (7.11) are purely imaginary when $\sqrt{B_{66}} < B_{12}$; $B_{66} > 0$ or complex when $\sqrt{B_{66}} > |B_{12}|$; $B_{66} > 0$, known as Type 1 and Type 2, respectively. Most orthotropic materials are categorized as type 1 and their conjugated roots are:

$$m_l = ip, \quad m_s = iq \quad (7.14)$$

where

$$p = \sqrt{B_{12} - \sqrt{B_{12}^2 - B_{66}}} \quad (7.15)$$

$$q = \sqrt{B_{12} + \sqrt{B_{12}^2 - B_{66}}} \quad (7.16)$$

The analytical solutions for asymptotic displacement and stress fields around a two dimensional traction-free interface crack tip were previously obtained by Lee (2000) and reported by Esna Ashari and Mohammadi (2012). The general crack tip displacement fields under the mixed mode loading in the upper layer ($k = 1$) of a cracked bimaterial specimen are (Lee, 2000):

$$\begin{aligned}
u_x = & \frac{K_I}{\sqrt{2\pi}(1+4\varepsilon^2)D \cosh(\varepsilon\tau)} \left\{ \right. \\
& e^{\varepsilon(\pi-\theta_l)} p_l \bar{A} \left[\cos\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) + 2\varepsilon \sin\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right] (r_l)^{\frac{1}{2}} \\
& + e^{-\varepsilon(\pi-\theta_l)} p_l A \left[\cos\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) + 2\varepsilon \sin\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) \right] (r_l)^{\frac{1}{2}} \\
& - e^{\varepsilon(\pi-\theta_s)} p_s \bar{B} \left[\cos\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) + 2\varepsilon \sin\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right] (r_s)^{\frac{1}{2}} \\
& - e^{-\varepsilon(\pi-\theta_s)} p_s B \left[\cos\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) + 2\varepsilon \sin\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) \right] (r_s)^{\frac{1}{2}} \left. \vphantom{\frac{K_I}{\sqrt{2\pi}(1+4\varepsilon^2)D \cosh(\varepsilon\tau)}} \right\} \\
& + \frac{K_{II}}{\sqrt{2\pi}(1+4\varepsilon^2)D \cosh(\varepsilon\tau)} \left\{ \right. \\
& - e^{\varepsilon(\pi-\theta_l)} p_l \bar{A} \left[\sin\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) - 2\varepsilon \cos\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right] (r_l)^{\frac{1}{2}} \\
& - e^{-\varepsilon(\pi-\theta_l)} p_l A \left[\sin\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) - 2\varepsilon \cos\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) \right] (r_l)^{\frac{1}{2}} \\
& + e^{\varepsilon(\pi-\theta_s)} p_s \bar{B} \left[\sin\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) - 2\varepsilon \cos\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right] (r_s)^{\frac{1}{2}} \\
& + e^{-\varepsilon(\pi-\theta_s)} p_s B \left[\sin\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) - 2\varepsilon \cos\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) \right] (r_s)^{\frac{1}{2}} \left. \vphantom{\frac{K_{II}}{\sqrt{2\pi}(1+4\varepsilon^2)D \cosh(\varepsilon\tau)}} \right\}
\end{aligned} \tag{7.17}$$

$$\begin{aligned}
u_y = & \frac{K_I}{\sqrt{2\pi}(1+4\varepsilon^2)D \cosh(\varepsilon\tau)} \left\{ \right. \\
& e^{\varepsilon(\pi-\theta_l)} q_l \bar{A} \left[\sin\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) - 2\varepsilon \cos\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right] (r_l)^{\frac{1}{2}} \\
& - e^{-\varepsilon(\pi-\theta_l)} q_l A \left[\sin\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) - 2\varepsilon \cos\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) \right] (r_l)^{\frac{1}{2}} \\
& - e^{\varepsilon(\pi-\theta_s)} q_s \bar{B} \left[\sin\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) - 2\varepsilon \cos\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right] (r_s)^{\frac{1}{2}} \\
& + e^{-\varepsilon(\pi-\theta_s)} q_s B \left[\sin\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) - 2\varepsilon \cos\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) \right] (r_s)^{\frac{1}{2}} \left. \vphantom{\frac{K_I}{\sqrt{2\pi}(1+4\varepsilon^2)D \cosh(\varepsilon\tau)}} \right\} \\
& + \frac{K_{II}}{\sqrt{2\pi}(1+4\varepsilon^2)D \cosh(\varepsilon\tau)} \left\{ \right. \\
& e^{\varepsilon(\pi-\theta_l)} q_l \bar{A} \left[\cos\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) + 2\varepsilon \sin\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right] (r_l)^{\frac{1}{2}} \\
& - e^{-\varepsilon(\pi-\theta_l)} q_l A \left[\cos\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) + 2\varepsilon \sin\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) \right] (r_l)^{\frac{1}{2}} \\
& - e^{\varepsilon(\pi-\theta_s)} q_s \bar{B} \left[\cos\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) + 2\varepsilon \sin\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right] (r_s)^{\frac{1}{2}} \\
& + e^{-\varepsilon(\pi-\theta_s)} q_s B \left[\cos\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) + 2\varepsilon \sin\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) \right] (r_s)^{\frac{1}{2}} \left. \vphantom{\frac{K_{II}}{\sqrt{2\pi}(1+4\varepsilon^2)D \cosh(\varepsilon\tau)}} \right\}
\end{aligned} \tag{7.18}$$

and the associated stress fields are:

$$\begin{aligned}
\sigma_x = & \frac{K_I}{2\sqrt{2\pi}D \cosh(\varepsilon\tau)} \left[\right. \\
& - p^2 \left\{ e^{\varepsilon(\pi-\theta_l)} \bar{A} \cos\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) + e^{-\varepsilon(\pi-\theta_l)} A \cos\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right\} (r_l)^{-\frac{1}{2}} \\
& + q^2 \left\{ e^{\varepsilon(\pi-\theta_s)} \bar{B} \cos\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) + e^{-\varepsilon(\pi-\theta_s)} B \cos\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right\} (r_s)^{-\frac{1}{2}} \left. \vphantom{\frac{K_I}{2\sqrt{2\pi}D \cosh(\varepsilon\tau)}} \right] \\
& + \frac{K_{II}}{2\sqrt{2\pi}D \cosh(\varepsilon\tau)} \left[\right. \\
& p^2 \left\{ e^{\varepsilon(\pi-\theta_l)} \bar{A} \sin\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) + e^{-\varepsilon(\pi-\theta_l)} A \sin\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right\} (r_l)^{-\frac{1}{2}} \\
& - q^2 \left\{ e^{\varepsilon(\pi-\theta_s)} \bar{B} \sin\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) + e^{-\varepsilon(\pi-\theta_s)} B \sin\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right\} (r_s)^{-\frac{1}{2}} \left. \vphantom{\frac{K_{II}}{2\sqrt{2\pi}D \cosh(\varepsilon\tau)}} \right]
\end{aligned} \tag{7.19}$$

$$\begin{aligned}
\sigma_y = & \frac{K_I}{2\sqrt{2\pi}D \cosh(\varepsilon\tau)} \left[\right. \\
& \left\{ e^{\varepsilon(\pi-\theta_l)} \bar{A} \cos\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) + e^{-\varepsilon(\pi-\theta_l)} A \cos\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right\} (r_l)^{-\frac{1}{2}} \\
& - \left\{ e^{\varepsilon(\pi-\theta_s)} \bar{B} \cos\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) + e^{-\varepsilon(\pi-\theta_s)} B \cos\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right\} (r_s)^{-\frac{1}{2}} \right] \\
& + \frac{K_{II}}{2\sqrt{2\pi}D \cosh(\varepsilon\tau)} \left[\right. \\
& \left\{ -e^{\varepsilon(\pi-\theta_l)} \bar{A} \sin\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) - e^{-\varepsilon(\pi-\theta_l)} A \sin\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right\} (r_l)^{-\frac{1}{2}} \\
& + \left\{ e^{\varepsilon(\pi-\theta_s)} \bar{B} \sin\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) + e^{-\varepsilon(\pi-\theta_s)} B \sin\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right\} (r_s)^{-\frac{1}{2}} \right]
\end{aligned} \tag{7.20}$$

$$\begin{aligned}
\sigma_{xy} = & \frac{K_I}{2\sqrt{2\pi}D \cosh(\varepsilon\tau)} \left[\right. \\
& \alpha_l \left\{ e^{\varepsilon(\pi-\theta_l)} \bar{A} \sin\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) - e^{-\varepsilon(\pi-\theta_l)} A \sin\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right\} (r_l)^{-\frac{1}{2}} \\
& + \alpha_s \left\{ -e^{\varepsilon(\pi-\theta_s)} \bar{B} \sin\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) + e^{-\varepsilon(\pi-\theta_s)} B \sin\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right\} (r_s)^{-\frac{1}{2}} \right] \\
& + \frac{K_{II}}{2\sqrt{2\pi}D \cosh(\varepsilon\tau)} \left[\right. \\
& \alpha_l \left\{ e^{\varepsilon(\pi-\theta_l)} \bar{A} \cos\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) - e^{-\varepsilon(\pi-\theta_l)} A \cos\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \right\} (r_l)^{-\frac{1}{2}} \\
& + \alpha_s \left\{ -e^{\varepsilon(\pi-\theta_s)} \bar{B} \cos\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) + e^{-\varepsilon(\pi-\theta_s)} B \cos\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \right\} (r_s)^{-\frac{1}{2}} \right]
\end{aligned} \tag{7.21}$$

where ε is the index of oscillation defined in (7.8) but now in terms of the second Dundurs parameter β for orthotropic materials (Dundurs, 1969)

$$\beta = \frac{h_{11}}{\sqrt{h_{12}h_{21}}} \tag{7.22}$$

$$\begin{cases} h_{11} = (l_{11})_1 - (l_{11})_2 \\ h_{21} = (l_{21})_1 + (l_{21})_2 \\ h_{12} = (l_{12})_1 + (l_{12})_2 \end{cases} \tag{7.23}$$

and

$$\begin{cases} (l_{11})_k = \left\{ \frac{p_s p - p_l q}{q - p} \right\}_k = \left\{ \frac{q_s - q_l}{q - p} \right\}_k \\ (l_{21})_k = \left\{ \frac{q q_l - p q_s}{q - p} \right\}_k \\ (l_{12})_k = \left\{ \frac{p_l - p_s}{q - p} \right\}_k \end{cases}, k = 1, 2 \quad (7.24)$$

Subscripts $k = 1, 2$ denote the upper and lower materials, respectively. Also,

$$p_l = -p^2 c_{11} + c_{12}, \quad p_s = -q^2 c_{11} + c_{12} \quad (7.25)$$

$$q_l = \frac{-p^2 c_{12} + c_{22}}{p}, \quad q_s = \frac{-q^2 c_{12} + c_{22}}{q} \quad (7.26)$$

$$A = q + \eta, \quad \bar{A} = q - \eta, \quad B = p + \eta, \quad \bar{B} = p - \eta, \quad D = q - p \quad (7.27)$$

and

$$\eta = \left(\frac{h_{21}}{h_{12}} \right)^{\frac{1}{2}} \quad (7.28)$$

Variables r_j, θ_j ($j = l, s$) are related to polar coordinate system (r, θ) as

$$r_j = r \sqrt{\cos^2 \theta + Z_j^2 \sin^2 \theta}, \quad j = l, s, \quad \begin{cases} Z_l = p \\ Z_s = q \end{cases} \quad (7.29)$$

$$\theta_j = \tan^{-1}(Z_j \tan \theta), \quad j = l, s, \quad \begin{cases} Z_l = p \\ Z_s = q \end{cases} \quad (7.30)$$

In order to obtain the displacement and stress fields in the material below the interface ($k = 2$), parameter $\varepsilon\pi$ is changed to $-\varepsilon\pi$.

Equations (7.17) and (7.18) will be used to derive the XFEM enrichment functions for reproduction of the singular orthotropic bimaterial interlaminar crack tip stress fields.

7.2.3 Stress contours for a crack between two dissimilar orthotropic materials

In order to further examine the analytical solutions for typical bimaterials, a sample plane stress problem of a crack between dissimilar orthotropic layers assumed. Materials axes of orthotropy are presumed to be parallel to y and x axes in materials 1 and 2, respectively, and the crack is assumed to extend from $(-5, 0)$ to $(0, 0)$ with the crack tip located at $(0, 0)$. The following material properties are used:

$$E_1 = 11.84 \text{ GPa}, E_2 = 0.81 \text{ GPa}, G_{12} = 0.63 \text{ GPa}, \nu_{12} = 0.38 \quad (7.32)$$

where the 1 and 2 subscripts denote the local material coordinates. In order to illustrate a graphical representation of the stress fields, the stress intensity factors K_I and K_{II} are assumed to be

$$K_I = 10, K_{II} = 10 \quad (7.31)$$

Fig. 7.2 illustrates the contours of the stress field σ_{xx} , σ_{yy} and σ_{xy} based on equations (7.19)-(7.21). The closest data point to the crack tip for drawing the contours was at 0.025 cm. It is clearly observed that while σ_{yy} and σ_{xy} remain zero along the traction free crack length, they remain continuous across the interface of two materials. Expectedly, σ_{xx} is not continuous along the crack length nor across the bimaterial interface.

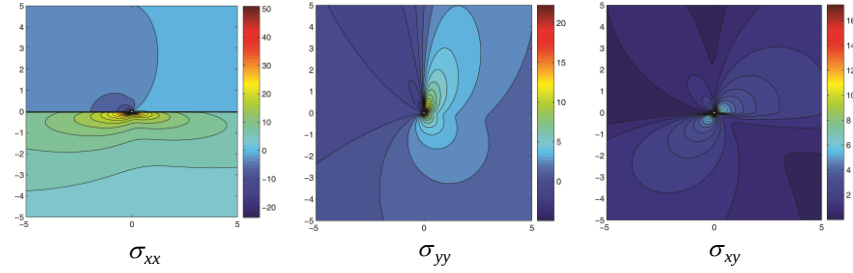


Figure 7.2 Analytical stress fields for an edge crack between two dissimilar orthotropic materials (Esna Ashari and Mohammadi, 2011a).

7.3 STRESS INTENSITY FACTORS FOR INTERLAMINAR CRACKS

The domain integral method can be utilized to evaluate mixed-mode stress intensity factors for interfacial cracks between orthotropic materials (Chow *et al.* 1995, Li *et al.*, 1985):

$$J = \int_A (\sigma_{ij} u_{i,1} - w_s \delta_{1j}) q_{,j} dA \quad (7.33)$$

where w_s is the strain energy density, A is the interior area of Γ contour surrounding the crack-tip (Fig. 7.3), and q is a function smoothly varying from $q = 1$ at the crack-tip to $q = 0$ at the exterior boundary Γ .

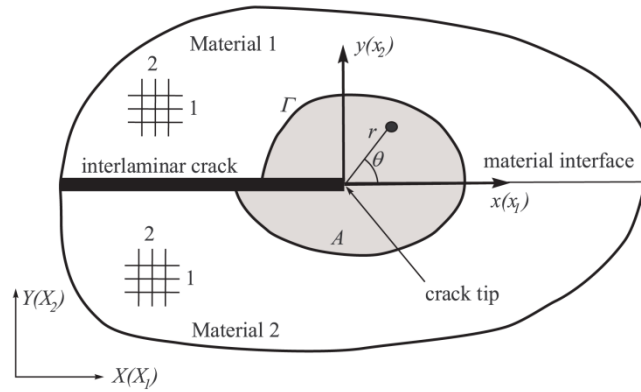


Figure 7.3 The contour Γ and its interior area, A .

In the interaction integral method, the auxiliary fields which should satisfy the boundary value problem (equilibrium equation and traction-free boundary condition on crack surfaces) are superimposed onto the actual fields to extract the mixed-mode stress intensity factors. One of the choices for the auxiliary state is the displacement and stress fields in the vicinity of the interfacial crack tip (7.17)-(7.21).

The J^S integral for the sum of the real and auxiliary states can be defined as:

$$J^S = J + J^{aux} + M \quad (7.34)$$

where J and J^{aux} are associated with the actual and auxiliary states, respectively, and M is the interaction integral, defined as

$$M = \int_A \left[\sigma_{ij} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - w^M \delta_{1j} \right] q_{,j} dA \quad (7.35)$$

Since the M integral shares the same path-independency of the original J integral, it can then be calculated away from the crack tip where the finite element solution is more accurate.

In order to determine the stress intensity factors of the present orthotropic bimaterial problem from the M integral, Chow *et al.* (1995) proposed the following relation to relate the stress intensity factors to the M integral:

$$K_i = 2 \sum_{m=1}^2 U_{im} M(u, u^{aux(m)}), i = 1, 2 \quad (7.36)$$

where K_I and K_{II} are obtained for the auxiliary states of ($K_I^{aux} = 1, K_{II}^{aux} = 0$) and ($K_I^{aux} = 0, K_{II}^{aux} = 1$), respectively, and

$$U = \left[(L_1^{-1} + L_2^{-1})^{-1} (I + \beta^2) \right]^{-1} \quad (7.37)$$

$$\beta = (L_1^{-1} + L_2^{-1})(S_1 L_1^{-1} - S_2 L_2^{-1}) \quad (7.38)$$

The subscripts 1 and 2 denote the upper and lower materials and S and L are 2×2 real Barnett and Lothe tensors, respectively (Barnett and Lothe, 1973). Explicit expressions of non-zero components of S and L for an orthotropic material have been derived by Dongye and Ting (1989) and Deng (1995),

$$S_{21} = \left[\frac{d_{66}(\sqrt{d_{11}d_{22}} - d_{12})}{d_{22}(d_{12} + 2d_{66} + \sqrt{d_{11}d_{22}})} \right]^{\frac{1}{2}} \quad (7.39)$$

$$S_{12} = -\sqrt{\frac{d_{22}}{d_{11}}} S_{21} \quad (7.40)$$

$$L_{11} = (d_{12} + \sqrt{d_{11}d_{22}}) S_{21} \quad (7.41)$$

$$L_{22} = \sqrt{\frac{d_{22}}{d_{11}}} L_{11} \quad (7.42)$$

where d_{ij} are the components of the contracted form of the fourth order elastic constant tensor $\mathbf{D}$, as defined in Eq. (4.10). Components of S and L are computed for each individual layer.

7.4 DELAMINATION PROPAGATION

7.4.1 Fracture energy based criteria

The state of stability of a 2D interlaminar crack can be examined by a mixed mode criterion in terms of the fracture energy release rate G and corresponding critical value G_c ,

$$\left(\frac{G_I}{G_{Ic}} \right)^{m/2} + \left(\frac{G_{II}}{G_{IIc}} \right)^{n/2} = 1 \quad (7.43)$$

Various values for parameters m and n have been proposed based on test data curve fitting techniques. For instance, while Borovkov *et al.* (1999) have proposed $m = 2.5, n = 3$, and Mi and Crisfield (1996) used $m = n = 4$, however, values $m = n = 2$ are generally adopted (Liu, 1994, Mi and Crisfield, 1996, Mohammadi *et al.*, 1998).

There is usually no need for any specific crack propagation criteria to determine the angle of crack propagation for 2D interlaminar cracks; predefined interface cracks are generally assumed to propagate along the interface. Therefore, once the stability of an

interface crack is violated, the interlaminar crack can be assumed to self-similarly propagate in a quasi-static manner. For dynamic problems, however, the extent of delamination can be determined from the crack tip velocity and the analysis time step. Asymptotic solutions for a moving orthotropic bimaterial crack has already been developed by Lee *et al.* (1996), which can be used to analyse dynamic delamination propagation in composites.

7.4.2 Stress based criteria

The simple Chang-Springer stress-based criterion has been frequently used for prediction of initiation or propagation of delamination in 3D problems (Chang and Springer, 1986),

$$\left(\frac{\sigma_n^2}{\bar{S}_n^2} \right) + \left(\frac{\sigma_{t1}^2 + \sigma_{t2}^2}{\bar{S}_t^2} \right) = 1 \quad (7.44)$$

where $\bar{S}_n$ and $\bar{S}_t$ are the unidirectional normal and tangential bonding strengths, respectively, σ_n represents the stress components normal to the interface and σ_{t1} and σ_{t2} are the two orthogonal tangential stress components on the tangential interface plane. The Hashin delamination criterion is based on the Chang-Springer approach combined with a linear softening law. The delamination function can be defined as,

$$F(\boldsymbol{\sigma}, \alpha) = f(\boldsymbol{\sigma}) - \bar{S}_n(\eta) \leq 1 \quad (7.45)$$

where

$$f(\boldsymbol{\sigma}) = \sqrt{\boldsymbol{\sigma}^T A \boldsymbol{\sigma}} \quad (7.46)$$

$$\bar{S}_n(\eta) = \bar{S}_{n0}(1 - \mu\eta) \quad (7.47)$$

with

$$A = \text{diag} \left[1, \frac{\bar{S}_n^2}{\bar{S}_t^2}, \frac{\bar{S}_n^2}{\bar{S}_t^2} \right] \quad (7.48)$$

η can be assumed as the equivalent inelastic strain and μ describes the slope of the softening curve $\bar{S}_n(\eta)$, which can be determined from the critical energy release rate G_c , the initial tensile strength $\bar{S}_{n0}$ and the characteristic thickness of the bonding layer h_t ,

$$\mu = \frac{\bar{S}_{n0} h_t}{2G_c} \quad (7.49)$$

Then, the rate of internal variable can be determined from an evolution law,

$$\dot{\eta} = -\dot{\lambda} \frac{\partial F(\boldsymbol{\sigma}, \eta)}{\partial \bar{S}_n} \quad (7.50)$$

where $\dot{\lambda}$ is the proportionality coefficient. For further details of the stress update procedure see Mohammadi *et al.* (2002) and Mohammadi and Moosavi (2007).

7.4.3 Contact based criteria

A number of delamination criteria have been proposed based on the use of a contact interaction approach for simulation of interlaminar behaviour (Mohammadi *et al.*, 1998, 1999, 2002, 2003). The Chang-Springer criterion can also be regarded in this category if σ_n and σ_t stress components are assumed as the contact interlaminar stresses. Assumptions for interaction between the normal and tangential fracture modes, especially in the softening regime, have resulted in a number of delamination criteria. Such an interaction is usually quantified by a variable k in terms of the ratios of normal and tangential relative displacements of the interface, g_n and g_t , respectively, (Tvergaard, 1990, Espinosa *et al.*, 2000)

$$k = \left\{ \left(\frac{g_n}{g_n^{\max}} \right)^\alpha + \left(\frac{g_t}{g_t^{\max}} \right)^\alpha \right\}^{\frac{1}{\alpha}} - 1 \quad (7.51)$$

where $g_n^{\max}$ and $g_t^{\max}$ are the maximum tolerable relative displacements in normal and tangential directions, respectively. α is a constant variable between 2 and 4 (usually 2).

The general methodology is usually designed on the basis of defining normal and tangential stress components of the interface for loading and unloading conditions based on the existing relative displacements g_n and g_t , maximum tolerable values of $g_n^{\max}$ and $g_t^{\max}$ and an assumption for the dependence of energy release rate G , variable k and the strength $\bar{S}_n$. For further details see Tvergaard (1990), Espinosa *et al.* (2000), Hashagen and de Borst (2000), Mi and Crisfield (1996), Mi *et al.* (1998), Mohammadi *et al.* (1998, 1999, 2002), Mohammadi and Forouzan-sepehr (2003) and Mohammadi and Moosavi (2007).

7.5 BIMATERIAL XFEM

The general methodology of orthotropic extended finite element can be similarly extended to include bimaterial problems, if bimaterial enrichment functions are embedded into an existing extended finite element framework. In addition to generalised Heaviside and bimaterial near crack tip enrichment functions, weak discontinuity enrichments are also required to guarantee continuity of the displacement field across the material interface. A variety of coupling terms between classical, Heaviside, crack tip and weak discontinuity degrees of freedom will form the final discretised stiffness matrix and the nodal force

vector.

7.5.1 Governing equation

Consider a bimaterial body in the state of equilibrium with the boundary conditions in the form of traction and displacement that also includes an interlaminar crack between the two parts of the constituents' orthotropic materials (Figure 7.4).

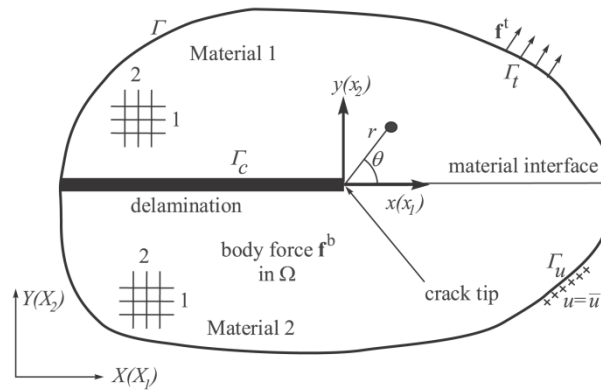


Figure 7.4 A body with an interface crack between two orthotropic materials.

The strong form of the equilibrium equation can be written as:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}^b = 0 \quad \text{in } \Omega \quad (7.52)$$

with the following boundary conditions:

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f}^t \quad \text{on } \Gamma_t : \text{external traction} \quad (7.53)$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u : \text{prescribed displacement} \quad (7.54)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_c : \text{traction free crack} \quad (7.55)$$

where $\mathbf{n}$ is normal to the boundary, Γ_t , Γ_u and Γ_c are traction, displacement and crack boundaries, respectively, $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{f}^b$ and $\mathbf{f}^t$ are the body force and external traction vectors, respectively.

The variational formulation of the boundary value problem can be defined as:

$$\int_{\Omega} \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \, d\Omega = \int_{\Omega} \mathbf{f}^b \cdot \delta \mathbf{u} \, d\Omega + \int_{\Gamma} \mathbf{f}^t \cdot \delta \mathbf{u} \, d\Gamma \quad (7.56)$$

7.5.2 XFEM discretisation

In order to model crack surfaces and crack tips in the extended finite element method, the approximate displacement function $\mathbf{u}^h$ can be expressed in terms of the classical displacement $\mathbf{u}$, the crack-split $\mathbf{u}^H$, the crack tip $\mathbf{u}^{tip}$ and the material interface $\mathbf{u}^{mat}$ components as

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{u}(\mathbf{x}) + \mathbf{u}^H(\mathbf{x}) + \mathbf{u}^{tip}(\mathbf{x}) + \mathbf{u}^{mat}(\mathbf{x}) \quad (7.57)$$

or more explicitly

$$\begin{aligned} \mathbf{u}^h(\mathbf{x}) = & \left[\sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j \right] + \left[\sum_{h=1}^{mh} N_h(\mathbf{x}) H(x) \mathbf{a}_h \right] \\ & + \left[\sum_{k=1}^{mt} N_k(\mathbf{x}) \left(\sum_{l=1}^{mf} F_l(x) \mathbf{b}_k^l \right) \right] + \left[\sum_{m=1}^{mm} N_m(\mathbf{x}) \chi_m(x) \mathbf{c}_m \right] \end{aligned} \quad (7.58)$$

and in a simple shifted form for H and F functions to guarantee the interpolation:

$$\begin{aligned} \mathbf{u}^h(\mathbf{x}) = & \left[\sum_{j=1}^n N_j(\mathbf{x}) \mathbf{u}_j \right] + \left[\sum_{h=1}^{mh} N_h(\mathbf{x}) (H(x) - H(x_h)) \mathbf{a}_h \right] \\ & + \left[\sum_{k=1}^{mt} N_k(\mathbf{x}) \left(\sum_{l=1}^{mf} [F_l(x) - F_l(x_k)] \mathbf{b}_k^l \right) \right] + \left[\sum_{m=1}^{mm} N_m(\mathbf{x}) \chi_m(x) \mathbf{c}_m \right] \end{aligned} \quad (7.59)$$

In Eqs. (7.58) and (7.59) n is the number of nodes of each finite element with classical degrees of freedom $\mathbf{u}_j$ and shape functions $N_j(\mathbf{x})$, mh is the number of nodes that have crack face (but not crack-tip) in their support domain, $\mathbf{a}_h$ is the vector of additional degrees of nodal freedom for modelling crack faces (not crack-tips) by the Heaviside function $H(\mathbf{x})$, mt is the number of nodes associated with the crack-tip in its influence domain, $\mathbf{b}_k^l$ is the vector of additional degrees of nodal freedom for modelling crack-tips, $F_l(\mathbf{x})$ are mf crack-tip enrichment functions, mm is the number of nodes that are directly affected by the weak discontinuity, $\mathbf{c}_m$ is the vector of additional degrees of nodal freedom for modelling weak discontinuity interfaces and $\chi_m(\mathbf{x})$ is the enrichment function (3.32) used for modelling weak discontinuities.

The first term in Eqs. (7.58) and (7.59) is the classical finite element approximation, the second and third terms are the enriched approximation related to crack surface and crack tip, respectively, while the last part is the enriched approximation used for modelling weak discontinuities. Fig. 7.5 illustrates these three types of enriched nodes in a finite element modelling of an interface crack. Other nodes and their associated finite element degrees of freedom are not affected by the presence of the crack.

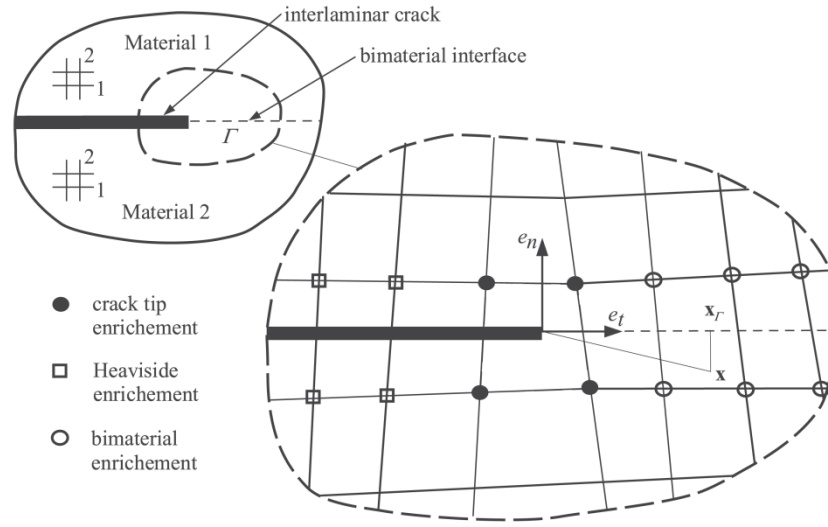


Figure 7.5 Node selections for enrichment; nodes enriched with crack-tip, Heaviside and weak discontinuity functions are marked by solid circles, squares and circles, respectively.

The discretised form of (7.56) based on XFEM approximation (7.59) can be written as a system of linear equations,

$$\mathbf{K}\mathbf{u}^h = \mathbf{f} \quad (7.60)$$

where $\mathbf{u}^h$ is the vector of nodal degrees of freedom for classical and enriched DOFs,

$$\mathbf{u}^h = \{\mathbf{u} \quad \mathbf{a} \quad \mathbf{b} \quad \mathbf{c}\}^T \quad (7.61)$$

$\mathbf{K}$ is the stiffness matrix and $\mathbf{f}$ is the vector of nodal force. The global stiffness matrix and force vector are calculated by assembling element stiffness matrix and force vector. $\mathbf{K}_{ij}$ for an element e is assembled from the following components:

$$\begin{cases} \mathbf{K}_{ij}^{rs} = \int_{\Omega^e} (\mathbf{B}_i^r)^T \mathbf{D} \mathbf{B}_j^s d\Omega \quad (r,s=u,a,b,c) \\ \mathbf{K}_{ij}^{ac} = \mathbf{K}_{ij}^{ca} = 0 \end{cases} \quad (7.62)$$

where $\mathbf{B}_i^u$, $\mathbf{B}_i^a$ and $\mathbf{B}_i^b$ are derivatives of shape functions defined in Eqs. (3.49)-(3.55), and no shifting procedure is required for derivatives of weak enrichment functions,

$$\mathbf{B}_i^c = \begin{bmatrix} (N_i\chi)_{,x} & 0 \\ 0 & (N_i\chi)_{,y} \\ (N_i\chi)_{,y} & (N_i\chi)_{,x} \end{bmatrix} \quad (7.63)$$

The vector of external forces $\mathbf{f}$ is defined as,

$$\mathbf{f}_i^e = \left\{ \mathbf{f}_i^u \quad \mathbf{f}_i^a \quad \mathbf{f}_i^b \quad \mathbf{f}_i^c \right\}^T \quad (7.64)$$

and

$$\mathbf{f}_i^u = \int_{\Gamma_t} N_i \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i \mathbf{f}^b d\Omega \quad (7.65)$$

$$\mathbf{f}_i^a = \int_{\Gamma_t} N_i H \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i H \mathbf{f}^b d\Omega \quad (7.66)$$

$$\mathbf{f}_i^{b\alpha} = \int_{\Gamma_t} N_i F_\alpha \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i F_\alpha \mathbf{f}^b d\Omega \quad (\alpha = 1, \dots, mf) \quad (7.67)$$

$$\mathbf{f}_i^c = \int_{\Gamma_t} N_i \chi \mathbf{f}^t d\Gamma + \int_{\Omega^e} N_i \chi \mathbf{f}^b d\Omega \quad (7.68)$$

Evaluation of (7.68) becomes important when a χ -enriched element (weak discontinuity or material interface) is subjected to a traction boundary condition.

7.5.3 XFEM enrichment functions for bimaterial problems

7.5.3.1 Orthotropic bimaterial enrichments

The analytical solutions (7.17) and (7.18) can be used to derive the necessary enrichment functions for an orthotropic bimaterial interface crack tip: (Esna Ashari and Mohammadi, 2011a)

$$\begin{aligned} \{F_l(r, \theta)\}_{l=1}^8 = & \left[e^{-\varepsilon\theta_l} \cos\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \sqrt{r_l}, \quad e^{-\varepsilon\theta_l} \sin\left(\varepsilon \ln(r_l) + \frac{\theta_l}{2}\right) \sqrt{r_l}, \right. \\ & e^{\varepsilon\theta_l} \cos\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) \sqrt{r_l}, \quad e^{\varepsilon\theta_l} \sin\left(\varepsilon \ln(r_l) - \frac{\theta_l}{2}\right) \sqrt{r_l}, \\ & e^{-\varepsilon\theta_s} \cos\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \sqrt{r_s}, \quad e^{-\varepsilon\theta_s} \sin\left(\varepsilon \ln(r_s) + \frac{\theta_s}{2}\right) \sqrt{r_s}, \\ & \left. e^{\varepsilon\theta_s} \cos\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) \sqrt{r_s}, \quad e^{\varepsilon\theta_s} \sin\left(\varepsilon \ln(r_s) - \frac{\theta_s}{2}\right) \sqrt{r_s} \right] \end{aligned} \quad (7.69)$$

with

$$r_j = r \sqrt{\cos^2 \theta + Z_j^2 \sin^2 \theta}, \quad j = l, s, \quad \begin{cases} Z_l = p \\ Z_s = q \end{cases} \quad (7.70)$$

$$\theta_j = \tan^{-1}(Z_j \tan \theta), \quad j = l, s, \quad \begin{cases} Z_l = p \\ Z_s = q \end{cases} \quad (7.71)$$

where all parameters have been defined in Section 7.2.2. These enrichment functions span the analytical asymptotic displacement fields for a traction free interfacial crack.

Nevertheless, it is important to note that the enrichment functions (7.69) correspond only to Type I of each composite layer, as defined by the complex roots of the characteristic Eq. (7.11). This is similar to the orthotropic enrichment functions (4.112), (4.115) and (4.116), previously proposed by Asadpoure *et al.* (2007) for crack analysis in single orthotropic materials.

7.5.3.2 Isotropic bimaterial enrichments

The orthotropic bimaterial enrichment functions (7.69) can be considered as an orthotropic extension to existing isotropic bimaterial enrichment functions, developed by Sukumar *et al.* (2004),

$$\begin{aligned} \{F_l(r, \theta)\}_{l=1}^{12} = & \left\{ \sqrt{r} \cos(\varepsilon \ln r) e^{-\varepsilon \theta} \sin \frac{\theta}{2}, \sqrt{r} \cos(\varepsilon \ln r) e^{-\varepsilon \theta} \cos \frac{\theta}{2}, \right. \\ & \sqrt{r} \cos(\varepsilon \ln r) e^{\varepsilon \theta} \sin \frac{\theta}{2}, \sqrt{r} \cos(\varepsilon \ln r) e^{\varepsilon \theta} \cos \frac{\theta}{2}, \\ & \sqrt{r} \sin(\varepsilon \ln r) e^{-\varepsilon \theta} \sin \frac{\theta}{2}, \sqrt{r} \sin(\varepsilon \ln r) e^{-\varepsilon \theta} \cos \frac{\theta}{2}, \\ & \sqrt{r} \sin(\varepsilon \ln r) e^{\varepsilon \theta} \sin \frac{\theta}{2}, \sqrt{r} \sin(\varepsilon \ln r) e^{\varepsilon \theta} \cos \frac{\theta}{2}, \\ & \sqrt{r} \cos(\varepsilon \ln r) e^{\varepsilon \theta} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \cos(\varepsilon \ln r) e^{\varepsilon \theta} \cos \frac{\theta}{2} \sin \theta, \\ & \left. \sqrt{r} \sin(\varepsilon \ln r) e^{\varepsilon \theta} \sin \frac{\theta}{2} \sin \theta, \sqrt{r} \sin(\varepsilon \ln r) e^{\varepsilon \theta} \cos \frac{\theta}{2} \sin \theta \right\} \end{aligned} \quad (7.72)$$

It is worth noting that functions (7.72) cannot be directly used for orthotropic bimaterial problems, because the oscillatory index ε is computed from isotropic material properties, which are not known for general orthotropic bimaterials, unless approximate average values are employed.

Finally, it should be emphasized that the general set of eight orthotropic bimaterial enrichment functions (7.69) contain all combined typical terms of $\sqrt{r_j} \cos(\theta_j/2)$ or $\sqrt{r}(\cos^2 \theta_j + Z^2 \sin^2 \theta_j)^{0.25} \cos(\theta_j/2)$. As a result, they also include all the twelve terms of $\sin \theta_j$, $\sin \theta_j \sin(\theta_j/2)$, $\sin \theta_j \cos(\theta_j/2)$, ... which are present in the set of isotropic enrichment functions (7.72). Therefore, the enrichment functions (7.69) generally span all various components of orthotropic and isotropic bimaterial enrichment functions.

7.5.3.3 Weak discontinuity enrichment

XFEM modelling of weak discontinuities has been comprehensively discussed in Section 3.6. They are used in places where the displacement field remains continuous while the strain field is discontinuous. The weak discontinuity enrichment function $\chi_m(\mathbf{x})$ can be defined as:

$$\chi_m(\mathbf{x}) = |\xi(\mathbf{x})| - |\xi(\mathbf{x}_m)| \quad (7.73)$$

where $\xi(\mathbf{x})$ is the signed distance function,

$$\xi(\mathbf{x}) = \min \|\mathbf{x} - \mathbf{x}_\Gamma\| \operatorname{sign}[\mathbf{e}_n \cdot (\mathbf{x} - \mathbf{x}_\Gamma)] \quad (7.74)$$

and $\mathbf{x}$, $\mathbf{x}_\Gamma$ ($\mathbf{x}_\Gamma \in \Gamma$) and $\mathbf{e}_n$ are defined in Fig. 7.5.

It should be noted that the weak discontinuity enrichment function is used only for the elements that are cut by the interface, avoiding generation of excessive errors in surrounding elements. Alternative functions are also available; e.g. Moës *et al.* (2003) have proposed a ridge enrichment function which is centered on the interface and has zero value on the elements not crossed by the interface.

7.5.4 Discretisation and integration

It is well known that the ordinary Gaussian rules do not accurately calculate the integration over the enriched elements cut by a crack or an interface. In order to improve the numerical integration procedure, a partitioning technique is adopted, which is based on subdividing such elements at both sides of the crack or interface into sufficient sub-triangles whose edges are adapted to crack faces or the interface, as illustrated in Fig. 7.6. In this partitioning approach, if the discontinuity is located very close to an edge, it might be better to avoid enriching that element to avoid numerical ill-conditioning and potential instabilities. For this purpose, if values of $A^-/(A^+ + A^-)$ or $A^+/(A^+ + A^-)$, where A^+ and A^- are the area of the influence domain of a node above and below the crack, respectively (see Fig. 7.6), are smaller than an allowable tolerance value, the node must not be enriched. The allowable tolerance value depends to each specific problem, but an average value of 0.01%, proposed by Dolbow *et al.* (2001), can be used.

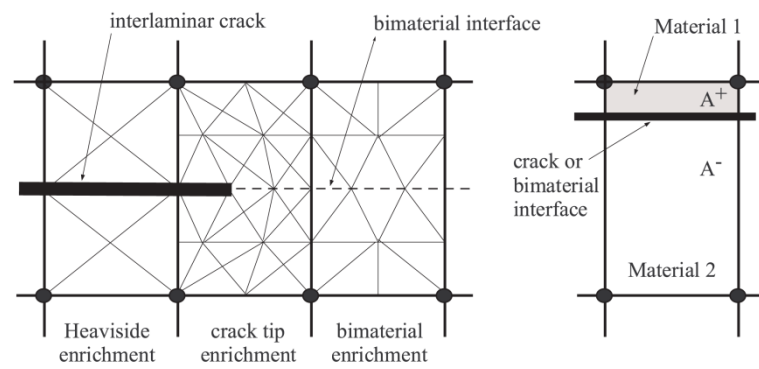


Figure 7.6 Cracked and bimaterial elements are subdivided into sub-triangles, and definition of A^+ and A^- .